

Proposal for an optical laser producing light at half the Josephson frequency

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We describe a superconducting device capable of producing laser light in the visible range at half of the Josephson generation frequency with the optical phase of the light locked to the superconducting phase difference. It consists of two single-level quantum dots embedded in a p - n semiconducting heterostructure and surrounded by a cavity supporting a resonant optical mode. We study decoherence and spontaneous switching in the device.

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Lasers and superconductors are both systems with macroscopic quantum coherence. In lasers, photons form a coherent state induced by stimulated emission of a driven system into a cavity mode. The resulting visible coherent light is characterized by an optical phase [1]. In superconductors, the ground state arising from spontaneous symmetry breaking is also characterized by a phase [2].

Traditionally, lasers and superconductors are studied separately. Recently [3], it has been realized that the superconducting (SC) phase difference and the optical phase may interact in a single device that combines two superconductors and a semiconducting p - n junction. The latter is a common system for light generation as the electron-hole recombination produces photons of visible frequency [4]. Combining semi- and superconductors within a nanostructure has been a difficult technological problem that attracted attention for a long time [5]. It has been solved using semiconductor nanowires [6] or quantum wells [7], opening up the possibility to make combined devices.

The device in question has been termed a Josephson LED [Fig. 1(a)]. It employs a double quantum dot (QD) in a p - n semiconductor nanowire connected to SC leads [3]. The device, biased with a voltage V , exhibits two types of photon emission: “blue” photons at the Josephson frequency $\omega_J = 2eV/\hbar$ due to the recombination of a Cooper pair from each side of the junction, and “red” photons at about $\omega_J/2$ due to electron-hole recombination. It has been shown that the optical phase of the Josephson generated “blue” photons is locked with the SC phase difference. The resulting “blue” light could in principle be enhanced by traditional optical methods but its small intensity makes this a challenging task.

In this Letter, we explore an alternative idea where the far more intense “red” emission is enhanced in a resonant cavity mode. We find lasing at half the Josephson frequency and, thus, dub the device ‘Half-Josephson Laser’ (HJL). In a common laser, lasing results from spontaneous symmetry breaking where all values of the optical phase are equivalent. Drift between these values leads to

a finite decoherence time. In contrast, the optical phase of the HJL is locked to the SC phase difference with only two allowed values of the optical phase corresponding to two opposite radiation amplitudes. This removes drift as a source of decoherence and opens up the possibility to manipulate the optical phase by changing the SC phase difference. Instead, decoherence of the radiation in the HJL results from switching between different QD states accompanied by the emission of a photon. We have explored these processes and find that by order of magnitude the resulting decoherence time is the same as the theoretical limit for a common laser $\tau_{\text{dec}} = n/\Gamma$, with Γ the damping rate and n the number of photons accumulated in the resonant mode. A rather low Γ is required to achieve lasing for a single Josephson LED, this condition being relaxed with a large number of LEDs in a single cavity [8].

Setup and model The HJL is a Josephson LED embedded in a single mode optical cavity with resonance

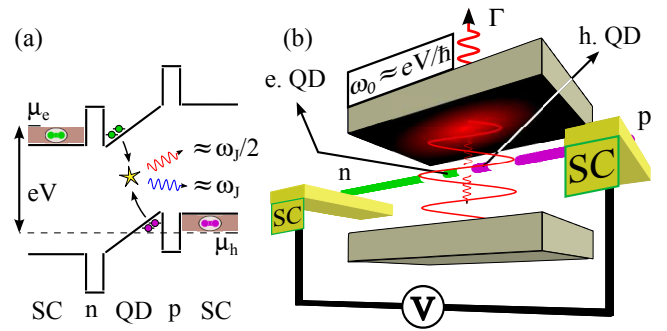


FIG. 1. (Color online) (a) The Josephson LED: electron (above) and hole (below) QD levels are close to the chemical potentials $\mu_{e,h}$ of the SC leads which differ by an energy eV . Charge transfer is only possible either through electron-hole recombination with the emission of a “red” photon at $\frac{1}{2}\omega_J$ or through a Cooper pair transfer with the emission of a “blue” photon at ω_J . (b) The HJL is a Josephson LED embedded in an optical cavity with a resonance frequency $\omega_0 \approx eV/\hbar$, i.e., close to the “red” emission frequency. The separately-colored regions in between the depleted areas represent the two QDs.

frequency $\omega_0 \approx \frac{1}{2}\omega_J$, Fig. 1(b). The light emission from the cavity is described by a damping rate Γ . The electronic part consists of a biased p - n junction where each side of the junction accommodates a QD connected to a SC lead. The barriers separating the QDs from the leads are arranged such as to allow charge transfer only through electron-hole recombination. Such QD junctions can be realized with semiconducting nanowires [9].

The minimal model for the QDs involves a single orbital for each QD. An orbital can house up to two particles (including spin) yielding 16 possible states. The QD Hamiltonian then reads [10]

$$\hat{H}_{\text{QD}} = \sum_{i=e,h} [E_i \hat{n}_i + U_i \hat{n}_i (\hat{n}_i - 1)] + U_{eh} \hat{n}_e \hat{n}_h, \quad (1)$$

where $\hat{n}_e = \sum_{\sigma} \hat{c}_{\sigma}^{\dagger} \hat{c}_{\sigma}$ ($\hat{n}_h = \sum_{\sigma} \hat{h}_{\sigma}^{\dagger} \hat{h}_{\sigma}$) is the electron (hole) number operator, and \hat{c}_{σ} (\hat{h}_{σ}) is the annihilation operator for an electron (hole) with spin σ . The energies $E_{e,h}$ are measured with respect to chemical potentials $\mu_{e,h}$ of the corresponding leads that differ by an energy $eV = \mu_e - \mu_h$; here, $U_{e,h} > 0$ is the on-site charging energy and $U_{eh} < 0$ the Coulomb attraction between electrons and holes. For concreteness, we assume that the hole level houses a heavy hole with $J_z = \pm \frac{3}{2}\hbar$, where z is the nanowire axis [4, 11]. Such levels are commonly used in optical experiments with QDs [12]. Our qualitative results do not depend on this particular choice.

Due to the proximity of the SC leads, Cooper pairs can coherently tunnel between the SC leads and the QDs introducing mixing between unoccupied and doubly occupied QD states. These processes can be compactly described by an additional term $\hat{H}_{\text{SC}} = \tilde{\Delta}_e^* \hat{c}_{\uparrow} \hat{c}_{\downarrow} + \tilde{\Delta}_h \hat{h}_{\uparrow} \hat{h}_{\downarrow} + \text{H.c.}$ in the Hamiltonian; here, the induced pair potentials $\tilde{\Delta}_{e,h}$ have reduced magnitudes in comparison with the gaps $\Delta_{e,h}$ of the SC leads but they retain the same phases $\phi_{e,h}$. Owing to gauge invariance, the physical quantities depend only on SC phase difference $\phi \equiv \phi_e - \phi_h$. The Hamiltonian is valid under the conditions $|\tilde{\Delta}_{e,h}|, E_{e,h}, U_{e,h}, U_{eh} \lesssim |\Delta|$. We note further that this Hamiltonian along with the electron-hole recombination conserves parity (*even* or *odd*) of the total number of particles on the QDs. Even-odd transitions require creation of quasiparticle excitations in the SC leads and occur with a relatively slow rate estimated below.

Interaction between the resonant mode and QDs is described by $\hat{H}_{\text{int}} = -\mathbf{E} \cdot \hat{\mathbf{d}}$; \mathbf{E} being electric field of the mode at QD position and $\hat{\mathbf{d}}$ the dipole moment of the optical transition between the conduction and the valence band. We assume a linear polarized mode, choose the x -axis in the direction of the polarization, and notice that for heavy holes $\hat{d}_x \propto (\hat{x} e^{-ieVt/\hbar} + \text{H.c.})$ with $\hat{x} \equiv (\hat{h}_{\downarrow} \hat{c}_{\uparrow} + \hat{h}_{\uparrow} \hat{c}_{\downarrow})$. The time-dependence of the dipole moment is due to the applied voltage. It is convenient to implement a rotating-wave approximation transferring the time-dependent factor to the photon creation (annihilation) operator \hat{b}^{\dagger} (\hat{b}). Thereby, the photon-dependent part of the Hamiltonian reads

$$\hat{H}_{\text{ph}} = \hbar\omega \hat{b}^{\dagger} \hat{b} + G(\hat{b}^{\dagger} \hat{x} + \hat{b} \hat{x}^{\dagger}) \quad (2)$$

with ω being the frequency *detuning*, $\omega = \omega_0 - eV/\hbar$, $|\omega| \ll \frac{1}{2}\omega_J$. We see that \hat{x} plays the role of a driving force that excites the oscillations in the mode. We note that all Hamiltonians considered conserve spin.

Semiclassics The present model is a rather complex case of nonequilibrium dissipative quantum mechanics. However, since we envisage a large number of photons in the mode, we employ a semiclassical approximation replacing $\hat{b} \mapsto \langle \hat{b} \rangle \equiv \lambda/G$. The Hamiltonian $H_{\text{QD}} + H_{\text{SC}} + H_{\text{ph}}$ can then be diagonalized to obtain the spectrum $E_m(\lambda)$ and corresponding eigenstates $|m\rangle$. The dipole strength $x_m(\lambda) \equiv \langle m | \hat{x} | m \rangle$ depends both on the radiation field λ and the QD state $|m\rangle$. Since the dipole strength in turn determines the evolution of the radiation field via the evolution equation

$$\dot{\lambda} = -\left(i\omega + \frac{\Gamma}{2}\right)\lambda - i\frac{G^2}{\hbar}x_m(\lambda), \quad x_m = \frac{\partial E_m}{\partial \lambda^*}, \quad (3)$$

we have to solve the system self-consistently [1]. The radiation field can build up as long as the energy gain rate $2\hbar\omega_0(G^2/\hbar)\text{Im}[x_m(\lambda)/\lambda]$ due to the nanowire is greater than the energy loss rate $\hbar\omega_0\Gamma$. With increasing λ the energy gain saturates till a *stationary state of radiation* (SSR) with $\dot{\lambda} = 0$ is reached at a certain radiation amplitude λ_s .

In conventional lasers, the driving is due to a population inversion that originates from dissipative transitions in an open system. For the HJL, the SC drive is not dissipative by itself: only the emission of photons from the cavity is a dissipative process. The driving originates from coherent mixing of discrete quantum states due to the proximity of the QD to the SC leads without any population inversion. Thus, the driving mechanism of the HJL is very different from that of a conventional laser. This is why the information about the SC phase difference is preserved in the process of driving. The energy gain, including its sign, depends on the difference between ϕ and the phase of λ . Owing to this, the phase of λ_s of the SSR is locked to the SC phase difference. The SSRs of the HJL come in pairs $\pm\lambda_s$ which is very different from a conventional laser where only the magnitude $|\lambda_s|$ (photon number) is fixed. We give in [8] analytical solutions to Eq. (3) for a toy two-level model.

Scales Let us estimate the scales involved that are expected to yield lasing. To simplify, we assume all characteristics of the QD spectrum to be of the same energy scale E which is of the order $E \simeq |\tilde{\Delta}_{e,h}| \ll eV$. This assures optimal mixing of the QD states by superconductivity. In a lasing state, the radiation amplitude should noticeably contribute to the energies of the QD states. This requires $|\lambda| \simeq E$. Assuming $\omega \simeq \Gamma$ we estimate from Eq. (3) that this takes place at $G \simeq \sqrt{\hbar\Gamma E}$.

We will assume that G is always chosen to be of this scale. The number of photons is then estimated as $n \simeq |\lambda|^2/G^2 \simeq E/\hbar\Gamma$. The semiclassical approximation is thus justified provided Γ is sufficiently small, $\Gamma \ll E/\hbar$.

Lasing Despite the model being minimal, it contains ten parameters that affect the existence and characteristics of the SSRs. To find these characteristics, we need to evaluate the dipole moment in a given state at given λ , and can do it separately for the states of *odd* and *even* parity since they are not mixed by interactions. Additionally, the spin conservation splits the eight odd states into two equivalent groups of four corresponding to total spin $\pm\frac{1}{2}$. For the even states, only one of the four possible $|1_e 1_h\rangle$ states, $(\hat{h}_\downarrow^\dagger \hat{c}_\uparrow^\dagger + \hat{h}_\uparrow^\dagger \hat{c}_\downarrow^\dagger)|0\rangle$, couples to the field. Hence, we only need to consider five of the eight even states as the other three are dark.

With this, we demonstrate lasing as proof of concept by finding SSRs in the even states for QD parameters within the above estimated scales, $-E_e = E_h = \frac{1}{2}U_e = \frac{1}{2}U_h = -U_{eh} = \Delta_h \equiv E$ and $\Delta_e = 1.5E$, for wide regions in the space of detuning ω and coupling G , see Fig. 2. Note that each eigenstate $|m\rangle$ has a different dipole strength x_m such that the lasing threshold G_c [Fig. 2(a)] and the radiation amplitudes of the SSRs λ_s [Fig. 2(c) and (d)] depend on m . Figure 2(b) shows the number of photons upon crossing the lasing threshold for the state $|2\rangle$. In agreement with the estimations, n reaches the maximum $\simeq E/\hbar\Gamma$ at $G \simeq \sqrt{\hbar\Gamma E}$. We stress that the optical phase of the radiation amplitude in an SSR is not arbitrary but locked to the SC phase difference [Fig. 2(c)].

Switching In the above discussion, we have assumed the QD to stay in a certain eigenstate $|m\rangle$. In fact it does not: the finite spectral width of the mode enables switching between the eigenstates. As shown below, the switching events occur on a much longer timescale Γ_{sw}^{-1} than that of the relaxation of λ towards its stationary value, Γ^{-1} . This separation of timescales allows to consider the switching dynamics separately from the dynamics of the radiation amplitude.

Each switching event is accompanied by the emission of a photon with a frequency mismatch compensating the difference of energies between initial and final eigenstates, $\hbar\omega_k = E_f - E_i$. For switchings not altering the parity of the eigenstate, the rates Γ_{sw} can be evaluated using Fermi's Golden Rule that contains the effective density of photon states $\Gamma/\hbar\omega_k^2$ (the tail of a Lorentzian-shaped emission line of the resonant mode) and the square of the matrix element, $|\langle m_f | \hat{H}_{\text{ph}} | m_i \rangle|^2$, with $|m_{i(f)}\rangle$ denoting the initial (final) state. Thereby, the switching rate can be estimated as $\Gamma_{\text{sw}} \simeq \Gamma G^2/E^2 \simeq \Gamma/n \ll \Gamma$. The switching events are thus rare and the device stays in one of the SSRs between the events.

Switchings altering parity are even rarer as they require the excitation of a quasi-particle above the SC energy gap $|\Delta_{e,h}|$. The larger detuning of the off-resonant

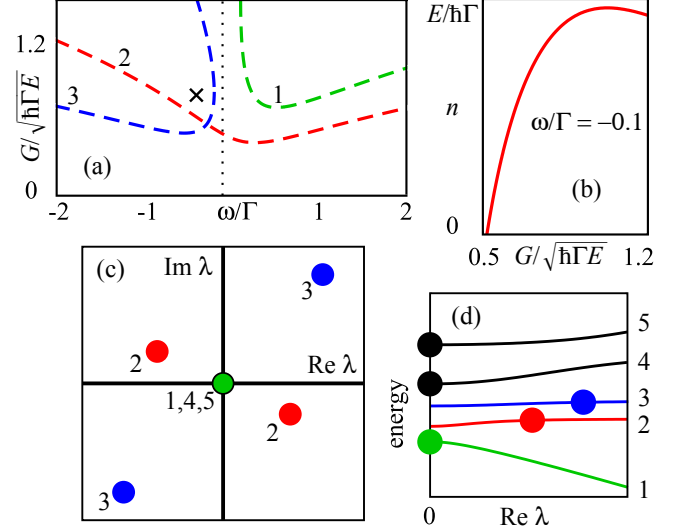


FIG. 2. (Color online) SSRs in the even states for QD parameters given in the text. The five eigenstates are labeled with numbers. (a) Lasing thresholds G_c for three eigenstates. At the line with the number m the energy gain at $\lambda = 0$ for the state $|m\rangle$ exactly equals the energy loss. (b) Number of photons n for eigenstate $|2\rangle$ versus the coupling constant G for $\omega/\Gamma = -0.1$ [dotted line in (a)] above the lasing threshold at $G = G_c \approx 0.5\sqrt{\hbar\Gamma E}$. Plots (c) and (d) illustrate the radiation amplitudes λ_s^m (marked with circles) for SSRs corresponding to the different eigenstates of the QD. The parameter choice is given by the cross in (a) where only the states $|2\rangle$ and $|3\rangle$ are lasing. (c) Non-zero λ_s^m come in pairs with opposite sign. Changing the SC phase difference will rotate the λ_s^m with respect to the origin of the plot. (d) Eigenenergies versus λ (at $\text{Im } \lambda = 0$). The λ_s^m are different for each $|m\rangle$.

photon $\omega_k \simeq |\Delta_{e,h}|/\hbar$ and an additional small factor $|\tilde{\Delta}_{e,h}/\Delta_{e,h}|$ result in a parametrically smaller rate $\Gamma_{e-o} \simeq |\tilde{\Delta}| \Gamma G^2 / |\Delta|^3 \ll \Gamma_{\text{sw}}$ [3]. Such processes do not conserve spin thus enabling switchings between dark and emitting states.

It is important to realize that, since the SSRs for different eigenstates have different values of λ_s^m , λ does not jump to the new stationary value upon a switching. Rather, the amplitude will evolve to $\lambda_s^{m_f}$ within a timescale $\simeq \Gamma^{-1}$, according to Eq. (3). For the same reason a switching event always involves different eigenstates rather than different SSRs at the same eigenstate. The latter would require large fluctuations of λ that are suppressed exponentially. Fig. 3 shows a sketch of the radiation intensity as function of time. In contrast to common lasers, the HJL intensity fluctuations are large at timescales of Γ_{sw}^{-1} .

Decoherence The intrinsic mechanism of decoherence in common lasers is a drift of the optical phase. For the HJL, this mechanism does not work since the amplitudes of the SSRs are locked to the SC phase difference. This renders switching the most important source of decoherence in the HJL. Indeed, after switching from a lasing

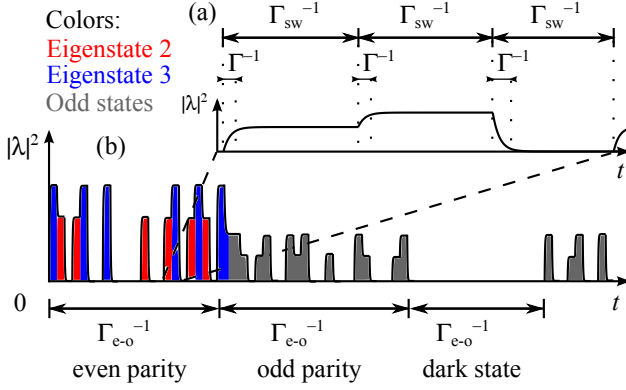


FIG. 3. (Color online) Sketch of the radiation intensity $|\lambda|^2$ evolving in time. (a) Switching events not altering the parity of the QDs occur at the timescale $\simeq \Gamma_{\text{SW}}^{-1}$. After switching, the radiation amplitude attains its new stationary value at a timescale $\simeq \Gamma^{-1} \ll \Gamma_{\text{SW}}^{-1}$ during which the QD remains in the same eigenstate. (b) Switching events altering the parity occur at a longer timescale $\simeq \Gamma_{e-o}^{-1} \gg \Gamma_{\text{SW}}^{-1}$. These can change between the dark and lasing states.

to a non-lasing SSR the radiation extinguishes quickly and its phase is forgotten. Even if the next switching brings the system to a lasing eigenstate, the radiation will evolve from the initial $\lambda = 0$ to any of the two possible $\pm\lambda_s^m$, with equal probability. Since decoherence is due to switching, the relevant timescale is given by $\tau_{\text{dec}} \simeq \Gamma_{\text{SW}}^{-1} \simeq n/\Gamma$. Despite the very different decoherence mechanism, this estimation is the same as for the common laser [13].

Average power and current The intensity fluctuations due to switching self-average at timescale exceeding Γ_{e-o}^{-1} . The averaged characteristics are expressed in terms of the probabilities P_s^m to be in a SSR s that belongs to an eigenstate $|m\rangle$. Those are given by the stationary solution to the master equation of the switching dynamics, that is composed of the switching rates [8]. In terms of these probabilities, the average number of photons in the cavity is given by $\bar{n} = \sum_{m,s} P_s^m |\lambda_s^m|^2 / G^2$. The average emission power is proportional to the photon number, $W = \frac{1}{2} \hbar \omega_J \Gamma \bar{n}$. The same holds for current in the device: since the emission of each photon is accompanied by a charge transfer, it is given by $I = e \Gamma \bar{n} = W/V$. An elaborated example of the current/intensity dependences is provided in [8]. For the current-voltage characteristic, we find a rather complex structure beside a peak with a magnitude of the order eE/\hbar , that is concentrated in a narrow interval $\simeq \hbar\Gamma/e$ of voltages in the vicinity $eV = \hbar\omega_0$. In this structure, two types of discontinuities are present: (i) kinks marking the thresholds of lasing instability at $\lambda = 0$ (second-order transitions) and (ii) jumps signaling the appearance of a SSR with stationary radiation amplitude λ_s far from $\lambda = 0$ (first-order transitions). We observe a relatively high probability to remain in an SSR with a large photon number. It is

explained from the fact that eigenstates at large λ are close to eigenstates of \hat{x} . This suppresses off-diagonal dipole-matrix elements resulting in a suppressed rate of transitions from this state.

Feasibility To show the feasibility of the HJL, we present here the estimations with concrete numbers. The SC gaps $|\Delta_{e,h}|$ are typically $\simeq 1$ meV, so we can choose the QD energy scale $E \simeq 0.1$ meV. To estimate the dipole strength $G \simeq ea|\mathbf{E}_0|$, with $|\mathbf{E}_0| \simeq \sqrt{\hbar\omega_0/\text{Vol}}$ being the quantum fluctuation of the electric field in the mode, we assume the cavity volume $\text{Vol} \simeq \ell^3$ with the wavelength $\ell = 2\pi c/\omega_0 \simeq 600$ nm, and take $a \simeq 5$ Å for the atomic distance scale. This gives the maximum $G \simeq 0.1$ meV.

With these two values for E and G , the minimum damping rate required for lasing is $\Gamma \simeq G^2/\hbar E \simeq 10^{11}$ Hz, corresponding to quality factor $Q \simeq 10^3$ which is common for optical cavities. However, in this situation the number of photons $n \simeq 1$. This can be enhanced by increasing Q and simultaneous decreasing G so it remains $\simeq \sqrt{\hbar\Gamma E}$. For photonic crystal cavities [14] quality factors $Q \simeq 10^6$ [15] have been measured and $Q \simeq 10^8$ [16] have been theoretically predicted. This gives photon numbers $n \simeq 10^3$ and $n \simeq 10^5$, respectively. The estimations of the emitted power and current at the peak do not depend on the choice of Γ and are given by $W \simeq 10$ nW, $I \simeq 10$ nA. The requirements on Γ can be eased and n enhanced by putting many Josephson LEDs in the same cavity. Furthermore, this also increases the emission power W and the current I [8].

In conclusion, we have demonstrated the feasibility of generating coherent visible light at half the Josephson frequency in a SC nanodevice. The workings of the device resemble the spontaneous parametric down-conversion in nonlinear optics [17] with the superconductors playing the role of coherent optical input. The novel driving mechanism results in locking between optical phase and SC phase difference. The decoherence of the emitted light originates from the switchings between different quantum states of the device.

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Supplementary Material to ‘Proposal for an optical laser producing light at half the Josephson frequency’

This material consists of three parts. In the first part we show explicitly how to find stationary values λ_s^m of the SSRs from Eq. (3) using a simple toy two-level model of the QD. In the second part, we provide a detailed example of the averaged quantities, lasing intensity and current, in dependence on voltage as promised in the main text. Finally, we briefly consider a device made by placing many Josephson LEDs into the same optical cavity.

Toy two-state model

A presentational problem with the setup described in the main text, is a relatively large number of QD states even when using all conservation laws (4 states for the odd parity or 5 states for the even parity). To circumvent this problem and still discuss the essential properties of the self-consistency equation and the novel driving mechanism, let us consider a toy two-state model. The great advantage of this approach is that all calculations can be performed explicitly.

Two-level system For the toy model, we take one of the states to be $|1_e 1_h\rangle$, which is unaffected by the superconductivity in the leads. The other state is chosen as a superposition of two states with an even number of particles in each dot, $\cos\theta|0_e 0_h\rangle + \sin\theta e^{i\phi}|2_e 2_h\rangle$. These number states are mixed by the SC leads with an angle θ . The phase ϕ is the SC phase difference. The Hamiltonian $H = H_{\text{QD}} + \lambda^* \hat{x} + \lambda \hat{x}^\dagger$ in this subspace is of the form

$$\hat{H} = \begin{pmatrix} E & \cos\theta \lambda + \sin\theta e^{i\phi} \lambda^* \\ \cos\theta \lambda^* + \sin\theta e^{-i\phi} \lambda & -E \end{pmatrix}. \quad (4)$$

Here, $2E$ is the energy difference between the two states in the absence of radiation and $\lambda = G\langle\hat{b}\rangle$ is due to the dipole coupling to the resonant mode. Diagonalizing this Hamiltonian yields the eigenenergies

$$E_{\pm}(\lambda) = \pm \sqrt{E^2 + |\lambda|^2 + \sin(2\theta)\text{Re}[\lambda^2 e^{-i\phi}]}, \quad (5)$$

where ‘ \pm ’ labels the two eigenstates of Eq. (4). Note that for $\lambda = 0$ we have $|+\rangle = |1_e 1_h\rangle$ and $|-\rangle = \cos\theta|0_e 0_h\rangle + \sin\theta e^{i\phi}|2_e 2_h\rangle$. The average values of the dipole operator in the eigenstates are calculated as

$$x_{\pm}(\lambda) = \langle \pm | \hat{x} | \pm \rangle = \frac{\partial E_{\pm}(\lambda)}{\partial \lambda^*} = \pm \frac{\lambda + \sin(2\theta)e^{i\phi}\lambda^*}{2\sqrt{E^2 + |\lambda|^2 + \sin(2\theta)\text{Re}[\lambda^2 e^{-i\phi}]}}, \quad (6)$$

The dipole serves as a driving force in the self-consistency equation which for stationary states is given by

$$\left(i\omega + \frac{\Gamma}{2}\right) \lambda = -i \frac{G^2}{\hbar} x_{\pm}(\lambda). \quad (7)$$

As was noted in the main text, a laser field can develop when the energy gain rate $G_E = 2\hbar\omega_0(G^2/\hbar)\text{Im}[x_m(\lambda)/\lambda]$ due to the Josephson LED is larger than the energy loss rate $\hbar\omega_0\Gamma$. With increasing λ the energy gain will saturate because the denominator of Eq. (6) increases, until a *stationary state of radiation* (SSR), λ_s , is reached ($\dot{\lambda}_s = 0$) and steady state lasing is achieved. For this toy model the energy gain is given by

$$G_{E\pm} = \mp \frac{G^2}{\hbar} \frac{\sin(2\theta) \sin(2\chi - \phi)}{\sqrt{E^2 + |\lambda|^2 [1 + \sin(2\theta) \cos(2\chi - \phi)]}} \quad (8)$$

where χ is the optical phase, $\lambda = |\lambda|e^{i\chi}$.

Driving mechanism In the main text we have stressed the novelty of the HJL driving mechanism. In conventional lasers, the driving is due to a population inversion that originates from dissipative transitions in an open system. However, for the HJL the drive is not dissipative and the energy gain depends crucially on mixing of QD states by the induced SC gaps without any population inversion.

In our toy-model the essence of the driving comes about through the mixed state $\cos\theta|0_e 0_h\rangle + \sin\theta e^{i\phi}|2_e 2_h\rangle$ which renders both $\lambda^* \langle - | \hat{x} | + \rangle$ and $\lambda \langle + | \hat{x} | - \rangle$ non-zero. Therefore, irrespective of the eigenstate of the QD, the system can always decay to the other state and emit a photon. Without mixing ($\theta = 0, \pi$) this is not possible. The energy required for the decay is supplied by the bias. With increasing λ the values of the amplitudes decrease as the eigenstates $|+(-)\rangle$ will become increasingly more like the eigenstates of \hat{x} . Hence the driving saturates.

From the expression for the energy gain, we explicitly see the role of the SC phase difference, ϕ . First, the value of this phase determines the sign of the energy gain, and thus whether it acts as a gain or as an absorber of radiation. Second, the SSRs with radiation amplitudes, λ_s , depend on the specific combination of phases, $2\chi - \phi$. Hence, the SSR values of the optical phase must depend on the SC phase difference, which signifies a phase-lock between the optical phase and SC phase difference.

We note that, because of this specific combination of phases, there are always two optical phases possible since changing $\chi \rightarrow \chi + \pi$ yields an SSR with the same radiation magnitude $|\lambda_s|$ but opposite sign. Hence, the SSRs come in pairs $\pm\lambda_s$. This is in stark contrast to conventional lasers where the driving is independent of the optical phase and all phases occur with equal probability.

As a final remark concerning the drive we note that the dipole x_{\pm} completely saturates in the limit of $\lambda \gg E$, to

$$x_{\pm} \approx \pm e^{i\chi} \frac{1 + \sin(2\theta)e^{i\phi-2i\chi}}{2\sqrt{1 + \sin(2\theta)\cos(2\chi - \phi)}}.$$

In this limit the eigenstates of H are the eigenstates of \hat{x} so that the amplitudes $\langle -|\hat{x}|+ \rangle$ and $\langle +|\hat{x}|- \rangle$ vanish.

Stationary states of radiation To find the radiation amplitudes of the SSRs, $\lambda_s^{m=\pm}$, we need to solve the stationary self-consistency equation with the dipole of Eq. (6). Here we see by inspection that $\lambda = 0$ is always a stationary solution since the dipole is proportional to $|\lambda|$. It may be unstable however against small fluctuations in the mode. By performing stability analysis we distinguish between the two cases.

To find a SSR with nonvanishing radiation amplitude, we first divide both sides of Eq. (7) (taking dipole x_+) with its conjugate to derive expressions for the optical phase

$$e^{2i\gamma} = \frac{1 + 2i\omega/\Gamma}{1 - 2i\omega/\Gamma} = -\frac{1 + \sin(2\theta)e^{i\phi-2i\chi}}{1 + \sin(2\theta)e^{2i\chi-i\phi}}. \quad (9)$$

where $\gamma = \arctan(2\omega/\Gamma)$. This reduces to

$$\cos(2\chi - \phi + \gamma) = -\frac{\cos \gamma}{\sin(2\theta)}. \quad (10)$$

with extra condition $\text{sgn}[\sin(2\theta)\sin(2\chi - \phi)] = 1$. From this, we explicitly see that the optical phase χ may assume two equivalent values shifted by π and that it is locked to SC phase ϕ . Nontrivial solutions for the optical phase χ exist for $\omega < -\frac{1}{2}\Gamma|\cot 2\theta|$, i.e., only at negative detuning. For the ‘-’ eigenvalue, the situation is reversed in detuning: the lasing solutions exist only at positive ω with $\omega > \frac{1}{2}\Gamma|\cot 2\theta|$.

Now taking modulus square of both sides of (7), we arrive at an expression for the optical field magnitude

$$\frac{G^4}{\hbar^2(\omega^2 + \frac{1}{4}\Gamma^2)} = 4 \frac{E^2 + |\lambda|^2[1 + \sin(2\theta)\cos(2\chi - \phi)]}{1 + \sin^2(2\theta) + 2\sin(2\theta)\cos(2\chi - \phi)}. \quad (11)$$

Solving Eqs. (10) and (11) simultaneously yields all SSRs, $\lambda_s^{m=\pm}$.

From Eq. (11) we may derive expressions for the lasing thresholds as it implies that the lasing solutions with $|\lambda| > 0$ can only exist above some critical coupling

$$\frac{G_c^2}{\hbar} = \frac{2|E|\sqrt{\omega^2 + \frac{1}{4}\Gamma^2}}{\sqrt{1 + \sin^2(2\theta) + 2\sin(2\theta)\cos(2\chi - \phi)}} = \frac{2|E|}{\cos^2(2\theta)} \left[|\omega| - \sqrt{\sin^2(2\theta)\omega^2 - \frac{1}{4}\cos^2(2\theta)\Gamma^2} \right].$$

Stability analysis shows that the zero stationary solution becomes unstable at this critical coupling. Because the lasing field continuously starts to grow when crossing this threshold, we have the analog of a second-order phase transition. Interestingly, at yet higher coupling

$$\frac{G_{c2}^2}{\hbar} = \frac{2|E|}{\cos^2(2\theta)} \left[|\omega| + \sqrt{\sin^2(2\theta)\omega^2 - \frac{1}{4}\cos^2(2\theta)\Gamma^2} \right]$$

the zero solution becomes stable again and we encounter an analog of the first-order transition where the SSRs with nonvanishing radiation amplitudes still exist at the threshold. At the tricritical point

$$G = \sqrt{\frac{2|E|\hbar\Gamma}{|\sin(4\theta)|}}, \quad |\omega| = \frac{\Gamma}{2}|\cot 2\theta|,$$

the transitions of both orders coexist.

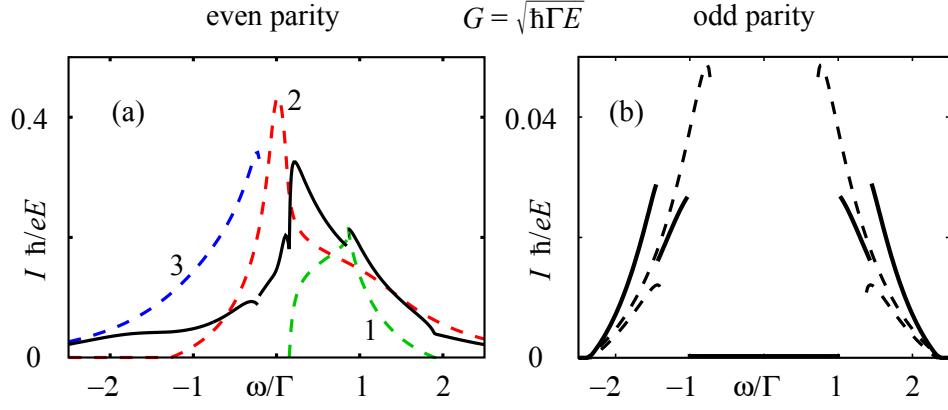


FIG. 4. (Color Online) Time-averaged current versus detuning for the even and the odd parity states. The dashed curves (reduced by factor of 3) give the current in individual lasing QD eigenstates, with the numbers in the even parity plot matching those of Fig. 2, in the main text. The solid curves represent the time-averaged currents which are the individual currents weighted with the probabilities P_s^m , the stationary solutions to master equation (12) of the switching dynamics as explained in the text. The QD parameters are as given in the main text, $-E_e = E_h = \frac{1}{2}U_e = \frac{1}{2}U_h = -U_{eh} = \Delta_h \equiv E$ and $\Delta_e = 1.5E$, and $G = \sqrt{\hbar\Gamma E}$.

Example of current/intensity dependence on voltage

In the main text the dynamics of the HJL over longer times was described in terms of switching rates. It was argued that these switchings occur after such long times that the corresponding dynamics is decoupled from the dynamics of the lasing field λ . Hence, the switchings always occur when the system is at an SSR of Eq. (3), λ_s^m , where the QD is in eigenstate $|m\rangle$. We additionally argued that (for $|\tilde{\Delta}| \ll |\Delta|$) switchings that change the parity of the QD state occur at an even slower rate. Even for $|\tilde{\Delta}| \simeq 0.1|\Delta|$ the parity switchings Γ_{e-o} are an order of 10^3 slower than the switchings Γ_{sw} . Therefore, when measuring over times much longer than Γ_{sw}^{-1} but shorter than Γ_{e-o}^{-1} , we see an average intensity corresponding to the SSRs of many QD eigenstates with equal parity. After a typical time scale Γ_{e-o}^{-1} the QD states switch parity and the average intensity corresponding to this parity is seen. The purpose of this section is to calculate the average intensity as a function of voltage for the even and odd parities separately and discuss some important features found in the corresponding curves.

The average intensity can be found by solving the master equation governing the switching dynamics and finding the corresponding stationary solution. The probability p_s^m to be at λ_s^m evolves in time according to

$$\frac{dp_s^m}{dt} = \sum_{n,r \neq m,s} [W_{m,s;n,r} p_r^n - W_{n,r;m,s} p_s^m] \equiv \sum_{n,r} \mathbf{W}_{m,s;n,r} p_r^n \quad (12)$$

where $W_{m,s;n,r}$ denotes the transition from $\lambda_r^n \mapsto \lambda_s^m$ and we define the transition matrix \mathbf{W} . The summation is over the stationary solutions λ_r^n of all eigenstates, $|n\rangle$. To solve master equation (12) we need to find and diagonalize \mathbf{W} . We notice that this matrix has a left eigenvector with zero eigenvalue because $\sum_{m,s} \mathbf{W}_{m,s;n,r} = 0$. Hence there must also be a right eigenvector with zero eigenvalue, which is the stationary distribution of probabilities ($\dot{p}_s^m = 0$ for all m) when normalized to unity. Since we are only interested in these stationary solutions we only need to find the right null vector of \mathbf{W} . We represent this stationary solution as P_s^m , with a capital P . Hence, we can find P_s^m when all $W_{m,s;n,r}$ are known.

To calculate the transitions $W_{m,s;n,r}$ we note that the switchings are accompanied by the emission of an off-resonant photon so that the rates follow from Fermi's Golden Rule. The transition rates are then proportional to the product of $|\langle m(\lambda_r^n) | \hat{H}_{ph} | n(\lambda_r^n) \rangle|^2$ and a Lorentzian shaped photon density of states, $\rho(\Delta E) = (1/2\pi)\hbar\Gamma/[(\hbar\Gamma/2)^2 + (\Delta E)^2]$, with $\Delta E = |E_n - E_m|$ the energy difference between $|n(\lambda_r^n)\rangle$ and $|m(\lambda_r^n)\rangle$. We note that the transitions go from eigenstate $|n(\lambda_r^n)\rangle \mapsto |m(\lambda_r^n)\rangle$ ($n \neq m$), which are the eigenstates of the Hamiltonian $H_{QD} + H_{sc} + H_{ph}$ in the semiclassical approximation, $\hat{b} \mapsto \langle \hat{b} \rangle \equiv \lambda/G$. Additionally we stress that the value of λ is unaltered during the transition, which is indicated by the (λ_r^n) in $\langle m(\lambda_r^n) |$ and $|n(\lambda_r^n)\rangle$. Only after the switch will λ evolve to λ_s^m according to Eq. (3).

To find the matrix elements of \hat{H}_{ph} [Eq. (2)] we only need to consider the interactions $G(\hat{b}^\dagger \hat{x} + \hat{b} \hat{x}^\dagger)$ as the term $\hbar\omega \hat{b}^\dagger \hat{b}$ will always yield zero contribution when $n \neq m$. The two interactions imply two possible transition amplitudes distinguishing two types of emission: one far below and one far above the cavity resonance. Since we are in the

rotating frame, this translates to creating a negative frequency photon with $\hat{b}\hat{x}^\dagger$, or creating a positive frequency photon with $\hat{b}^\dagger\hat{x}$. We do not expect absorption of off-resonant photons as they are very scarce. Taking the detuning into account, the photon energies are $\hbar\omega_k = \hbar\omega - \Delta E$ and $\hbar\omega_k = \hbar\omega + \Delta E$ respectively. The detuning may be neglected, however, as it is typically taken of the same order as $\Gamma \ll \Delta E$. For this same reason we may also neglect the Γ in the denominator of the density of states $\rho(\Delta E)$.

Taking the above considerations into account we find the transition rates

$$W_{m,s;n,r} = \frac{\Gamma G^2}{(\hbar\omega_k)^2} |\langle m(\lambda_r^n) | \hat{x} | n(\lambda_r^n) \rangle - \langle m(\lambda_r^n) | \hat{x}^\dagger | n(\lambda_r^n) \rangle|^2. \quad (13)$$

With this we can find \mathbf{W} and thus P_s^m . We make some brief remarks about \mathbf{W} . As stipulated above the switchings only occur between different QD eigenstates. Fluctuations that let the system jump from one to another SSR in the same QD eigenstate are exponentially small due to the large number of photons. Thus $W_{m,s;m,r} = 0$ if $s \neq r$. Furthermore, we note that when in a lasing eigenstate and starting at $\lambda = 0$ the system may evolve to both SSRs with $\pm\lambda_s^m$ with equal probability. Hence the transition rate from a non-lasing QD eigenstate to the SSR with λ_s^m of a lasing QD eigenstate is half the value calculated in Eq. (13). For transitions between two lasing eigenstates it must be determined to which of the multiple stationary points λ_s^m the field evolves. This is done by integrating the time evolution equation of λ , Eq. (3), until one of the λ_s^m is sufficiently approached, using the radiation amplitude λ_r^m of the previous SSR as initial condition.

With the stationary probability distribution, P_s^m , we calculate some measurable quantities. First, the average number of photons in the cavity is $\bar{n} = \sum_{m,s} P_s^m |\lambda_s^m|^2 / G^2$. Then the average number of photons that escape the cavity is $\Gamma\bar{n}$, yielding the average emission power $W = \frac{1}{2}\hbar\omega_J\Gamma\bar{n}$ and the average current in the device $I = e\Gamma\bar{n}$. The latter because every photon emission is accompanied by a single charge transfer.

Using the minimal model of the main text we calculate the current/intensity as a function of detuning. As mentioned above we plot the dependencies for odd and even parity states separately. The results are presented in Fig. 4. Since the detuning is related to the applied voltage via $\omega = eV/\hbar - \omega_0$, the peaks in the current, of the order of eE/\hbar , are in a narrow interval $\simeq \hbar\Gamma/e$ of voltages. There are two kinds of discontinuities in the curves: kinks, that mark the thresholds of the lasing instability at $\lambda = 0$ in some QD eigenstate (second-order transitions), and jumps, that signal the appearance of an SSR far from $\lambda = 0$ (first-order transitions). The curves also show a relatively high probability to be in lasing QD eigenstates if available. This can be qualitatively understood by considering that eigenstates at large λ are close to eigenstates of \hat{x} . This suppresses off-diagonal matrix elements, Eq. (13), resulting in a slower rate of transitions away from a lasing state. For the odd parity states in Fig. 4, the time-averaged current drops to zero while a lasing state is still available. This is because the non-lasing solution in this lasing eigenstate becomes stable at $|\omega|/\Gamma \lesssim 1$ and whereas all other states are nonlasing. The device then gets stuck in the SSRs at $\lambda = 0$ even though SSRs with $\lambda_s \neq 0$ are available.

Many Josephson LEDs in a single cavity

We have discussed in the main text that the large number of photons $n \simeq E/\hbar\Gamma$ (assuming $G \simeq \sqrt{\hbar\Gamma E}$) in a cavity with a single Josephson LED requires high Q factors. This requirement can be easily softened, and the lasing can be achieved at higher damping rate Γ , if we incorporate $N \gg 1$ Josephson LEDs in the cavity. Since the LEDs are of ten nanometer scale, one can put hundreds of them even into a single-wavelength cavity. The resulting system certainly deserves to be explored in detail, and will be a topic of separate research presented elsewhere. In this note we just give some straightforward estimations.

All N QD dipoles are in this setup coupled to a single mode, were we assume equal coupling coefficients G . The self-consistency equation for the radiation amplitude changes to

$$\dot{\lambda} = - \left(i\omega + \frac{\Gamma}{2} \right) \lambda - i \frac{G^2}{\hbar} \sum_i^N x_{m_i}(\lambda), \quad x_{m_i} = \frac{\partial E_{m_i}}{\partial \lambda^*}. \quad (14)$$

Assuming for simplicity all dipoles to be in the same direction with same or similar parameter values, we may simplify the sum over dipoles to Nx_m . This yields $\lambda \simeq G^2 N / \hbar\Gamma$ for the SSRs. The optimal regime near the lasing threshold is achieved if $\lambda \simeq E$. Thus this corresponds to $G \simeq \sqrt{\hbar\Gamma E / N}$. Therefore, the lasing threshold can be achieved at much larger damping rate $\Gamma \simeq G^2 N / \hbar E$.

This also increases the number of photons in the mode. If we optimize the coupling to $G \simeq \sqrt{\hbar\Gamma E / N}$, the number is estimated as $n \simeq NE/\hbar\Gamma$, N times larger than the estimate for a single-LED device. If we fix Γ , G and increase the

number of wires in the device, the photon number scales as $n \simeq N^2 G^2 / \Gamma^2$. Note that in both cases the emitted power increases as well. It is respectively given by $W \simeq \Gamma e V n \simeq e V N E / \hbar$, for optimal coupling, and $W \simeq e V N^2 G^2 / \Gamma$ when only increasing the number of wires.

It is likely that a large number of LEDs in the cavity greatly reduces the intensity fluctuations. With only one pair of QDs the lasing intensity fluctuates down to zero as can be seen from Fig. 3, in the main text. In contrast, the switching of only one of N QDs will hardly affect the intensity, so the relative intensity fluctuations are expected to be of the order of $1/\sqrt{N}$. Likewise, the phase of the optical field will be barely affected by the switching in one nanowire. Therefore the decoherence times may become significantly longer than in a single LED HJL.